About the Abel map computation

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Divisor of a Riemann surface

- $K = \mathbb{Q}(\alpha)$; $F \in K[X, Y]$ defines a Riemann surface of genus $g$. 

\[ D = \sum n_P P \] formal sum of points.

Degree of a divisor: $\deg(D) = \sum n_P$.

Function divisor: $\text{div}(f) = \sum v_P(f)$, $P \to$ zeros and poles of $f$. 

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Divisor of a Riemann surface

\( \mathbb{K} = \mathbb{Q}(\alpha) ; \ F \in \mathbb{K}[X, \ Y] \) defines a Riemann surface of genus \( g \).  

- Divisor \( D = \sum n_{\mathcal{P}} \mathcal{P} \) formal sum of points.
- Degree of a divisor: \( \deg(D) = \sum n_{\mathcal{P}} \)
- Function divisor \( (f) = \sum \nu_{\mathcal{P}}(f) \mathcal{P} \rightarrow \) zeros and poles of \( f \).
Is a degree 0 divisor a function divisor?

Let
- $\mathcal{F}$ the set of function divisors.
- $\mathcal{D}_0$ the set of degree 0 divisors.

We have $\mathcal{F} \subset \mathcal{D}_0$ but no equality. Thus the following questions:

- Let $D \in \mathcal{D}_0$. Does $D \in \mathcal{F}$?
- Is there $n \in \mathbb{Z}$ such that $nD \in \mathcal{F}$?
- Given $D_1, \ldots, D_p \in \mathcal{D}_0$, is there $n_1, \ldots, n_p \in \mathbb{Z}$ such that $n_1D_1 + \cdots + n_pD_p \in \mathcal{F}$?
One answer: Abel-Jacobi theorem

The Abel map

\[ A : \mathcal{D}_0 / \mathcal{F} \rightarrow \mathbb{C}^g / \Gamma \]

\[ \sum n_P P \mapsto \sum n_P \left( \int_P \omega_1, \ldots, \int_P \omega_g \right) \]

is a group isomorphism.

\[ \Rightarrow \text{Computing the Abel map answers our questions !} \]
Details on the Abel map

$$A : \mathcal{D}_0/F \rightarrow \mathbb{C}^g/\Gamma$$

$$\sum n_P P \mapsto \sum n_P \left( \int_P \omega_1, \cdots, \int_P \omega_g \right)$$

- There are $2g$ cycles that cannot be reduced to a point.
  $\rightarrow$ homology basis.

- Periods: integrals of holomorphic differentials over this basis.

- Modulo the periods, $\int_P \omega_i$ does not depend of the path choice.
Motivations

Computer algebra

- Integration (logarithmic part)
  - Risch 69, *The problem of integration in finite terms*
  - Bronstein 1990, *Integration of elementary functions*
  - Bertrand 95, *Computing a hyperelliptic integral using arithmetic in the jacobian of the curve*

- Symbolic ODEs
  - algebraic solutions,
  - Baldassari & Dwork 79, *On second order linear differential equations with algebraic solutions*
  - Galois group
  - Compoint & Singer 79, *Relations linéaires entre solutions d’équations différentielles*

Physics

- Quasi periodic solutions of the KP equation
  - Deconinck & Segur 1998, *The KP equation with quasiperiodic initial data*

- Special solutions of the KdV and NLS equations
  - Deconinck & Patterson 2007, *Computing the Abel map*
Computing the Abel map: overview

Patterson 2007, *Algebro-geometric algorithms for integrable systems*

1. Computing the period matrix

van Hoeij & Deconinck 2001, *Computing Riemann matrices of algebraic curves*

- Monodromy computation,
- Use it to compute the homology basis,
  \[\Rightarrow\] Tretkoff\(^2\) 1984, *Combinatorial Group Theory, Riemann Surfaces and Differential Equations.*
- Compute a basis of holomorphic differential on the surface,
- Compute the periods.

2. Make analytic continuation along the needed paths.
Regular and critical points: a projective point of view

\[ C = \{(x, y) \in \mathbb{C}^2 \mid F(x, y) = 0\} \]

Let \( x_0 \in \mathbb{C} \):

- **Fiber** at \( x_0 \):
  \[ F(x_0) = \{\text{roots of } F(x_0, Y) = 0\}. \]

- **Regular point** : \( \#F(x_0) = d_Y \).

- **Critical point** : \( \#F(x_0) < d_Y \).

\[ \implies \text{roots of } R_F = \text{Res}_Y(F, F_Y) \]
Analytic continuation: compute fibers and connect

1. Choice of paths,

2. Choice of connection points,
Analytic continuation: compute fibers and connect

1. Choice of paths,
2. Choice of connection points,
3. Compute the fibers,
Analytic continuation: compute fibers and connect

1. Choice of paths,
2. Choice of connection points,
3. Compute the fibers,

\[ e_i \rightarrow e_{i+1} \]
\[ y_1 \rightarrow y_2 \rightarrow y_3 \]
\[ z_1 \rightarrow z_2 \rightarrow z_3 \]

\[ \gamma \]

\[ X \]
Local monodromy: $F(X, Y) = Y^3 - X$ around $X = 0$

- analytic continuation around the critical point $X = 0$
  $\implies$ permutation on the fiber
Monodromy group

- Let \( \alpha_1, \ldots, \alpha_d \) denote the critical points.

- We fix a regular base point \( a \).

- We will compute the \( d \) permutations \( \sigma_1, \ldots, \sigma_d \) generated by \( \gamma_1, \ldots, \gamma_d \).

- These permutations generate the monodromy group.
Computing the Abel map: details

Patterson 2007, *Algebro-geometric algorithms for integrable systems*

1. Computing the period matrix

van Hoeij & Deconinck 2001, *Computing Riemann matrices of algebraic curves*

- Monodromy computation,
- Use it to compute the homology basis,
- Compute a basis of holomorphic differential on the surface,
- Compute the periods.

2. Make analytic continuation along the needed paths.

- Makes analytic continuation using first derivative only,
- Uses Puiseux expansion when $\mathcal{P}$ is close from a critical point.
Connexion method

- van Hoeij & Deconinck 2001
  - Connexion using only derivatives (order 1),
  - No certification that the connexion is correct.

- van Hoeij & Rybowicz 2006
  - Certified algorithm using order 1 truncations,
  - Efficiency problem: sometimes too many intermediary steps (two close critical points).

- Poteaux 2007, *Computing monodromy groups defined by plane algebraic curves*
  - Power series expansions at well chosen truncation orders,
  - Trade-off between truncations orders and the number of steps,
  - (similar to Chudnovsky\(^2\) 86, 87, 90, van der Hoeven 99 for ODEs)
  - *useful for the integration part.*
Choice of the paths

- Most methods: going as far as possible from the critical points (one loop = 17 steps)
  - van Hoeij & Deconinck 01, Chudnovsky\(^2\) 86, 87, 90, van der Hoeven 99
  - numerical problems when two critical points are very close...

- Another way: computing series \textit{above critical points}
  - van der Hoeven 01 (ODE) Poteaux 07 (Puiseux series)
  - one loop = one evaluation,
  - Puiseux series are also useful for the Abel map!
    - needed when the point \( \mathcal{P} \) is close to a critical point,
    - Deconinck-Patterson 2007, \textit{Computing the Abel map}
    - it enables to compute integral basis (thus cohomology).
      - van Hoeij 1994, \textit{an algorithm for computing an integral basis in an algebraic function field}
Puiseux series

- A generalization of power series:

  - above a regular point: \( Y(X) = \sum_{k=0}^{\infty} \beta_k (X - x_0)^k \)

  - above a critical point: \( Y(X) = \sum_{k=n}^{\infty} \beta_k (X - x_0)^{k/e} \)

- Costly to compute:
  - coefficient growth and high degree extension field
  - one example: \( (Y^3 - X)((Y - 1)^2 - X)(Y - 2 - X^2) + X^2 Y^5 \)
    \( \implies \) extension of degree 92, rational numbers with 132 digits
  - Bit complexity: \( O^\sim(dY^{32} dX^4 \text{ht}(F)^2) \) Walsh 2000
  - Best known arithmetic complexity: \( O^\sim(D^5) \) Poteaux 2008

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Computing Puiseux series efficiently

- A modular-numeric strategy to avoid coefficient growth:
  
  - First computation modulo a well-chosen prime number
    \[ \Rightarrow \text{structure of the solution}, \]
  
  - Using it, conduct numerical computations to get an approximation of the coefficients.

- Towards a faster algorithm
  
  - Computing fast “Taylor parts” + factorize \( F \) in \( \mathbb{K}[[X]][Y] \) during the process + using optimal truncation bounds via relaxed algorithms \( \Rightarrow \mathcal{O}(D^4) \)
  
  - Being able to factorize \( F = F_1 \cdot F_2 \) in \( \mathbb{K}[[X]][Y] \) knowing singular part of Puiseux series of \( F_1 \) \( \Rightarrow \mathcal{O}(D^3) \)
What to be done yet?

- Numerical part of Puiseux is not certified and might be improved,
- Fast Puiseux algorithm to be finished and written properly,
- Still one missing point for the $O^\sim(D^3)$ strategy,
- Efficient implementation of all these algorithms,
- Studying the next parts of the Abel map (optimizations, implementation...)

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towards a new strategy