About the Abel map computation

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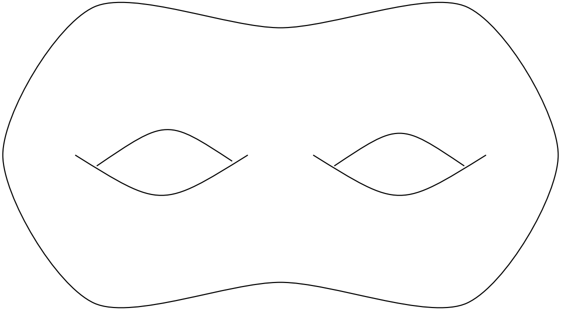
Divisor of a Riemann surface

- $\mathbb{K} = \mathbb{Q}(\alpha)$; $F \in \mathbb{K}[X, Y]$ defines a Riemann surface of genus g.



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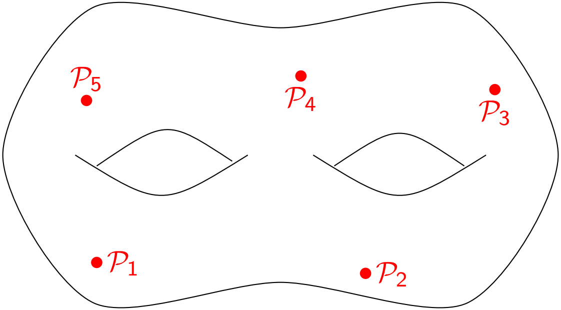


- Divisor $D = \sum n_{\mathcal{P}} \mathcal{P}$ formal sum of points.
- Degree of a divisor: $\deg(D) = \sum n_{\mathcal{P}}$
- Function divisor $(f) = \sum v_{\mathcal{P}}(f)\mathcal{P} \to$ zeros and poles of f.

Is a degree 0 divisor a function divisor ?

Let - $\mathcal{F}$ the set of function divisors.

- $\mathcal{D}_0$ the set of degree 0 divisors.

We have $\mathcal{F} \subset \mathcal{D}_0$ but no equality. Thus the following questions:

- Let $D \in \mathcal{D}_0$. Does $D \in \mathcal{F}$?
- Is there $n \in \mathbb{Z}$ such that $nD \in \mathcal{F}$?
- Given $D_1, \ldots, D_p \in \mathcal{D}_0$, is there $n_1, \ldots, n_p \in \mathbb{Z}$

  s.t. $n_1 D_1 + \cdots + n_p D_p \in \mathcal{F}$?

One answer: Abel-Jacobi theorem

Theorem

The Abel map

$$\begin{array}{rccc} \mathcal{A}: & \mathcal{D}_0/\mathcal{F} & \longrightarrow & \mathbb{C}^g/\Gamma \\ & \sum n_{\mathcal{P}}\mathcal{P} & \longmapsto & \sum n_{\mathcal{P}} \left(\int_{\mathcal{O}}^{\mathcal{P}} \omega_1, \cdots, \int_{\mathcal{O}}^{\mathcal{P}} \omega_g \right) \end{array}$$

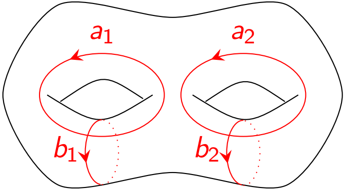
is a group isomorphism.

$\Longrightarrow$ Computing the Abel map answers our questions !

Details on the Abel map

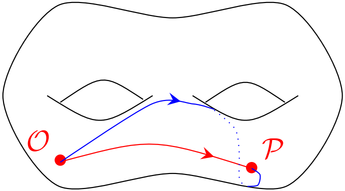
$$\begin{array}{rccc} \mathcal{A}: & \mathcal{D}_0/\mathcal{F} & \longrightarrow & \mathbb{C}^g/\Gamma \\ & \sum n_{\mathcal{P}}\mathcal{P} & \longmapsto & \sum n_{\mathcal{P}} \left(\int_{\mathcal{O}}^{\mathcal{P}} \omega_1, \cdots, \int_{\mathcal{O}}^{\mathcal{P}} \omega_g \right) \end{array}$$

- There are $2\,g$ cycles that cannot be reduced to a point.
  $\Longrightarrow$ homology basis.



- Periods: integrals of holomorphic differentials over this basis.

- Modulo the periods, $\int_{\mathcal{O}}^{\mathcal{P}} \omega_i$ does not depend of the path choice.



Motivations

Computer algebra

- Integration (logarithmic part)

Risch 69, *The problem of integration in finite terms*
Bronstein 1990, *Integration of elementary functions*
Bertrand 95, *Computing a hyperelliptic integral using arithmetic in the jacobian of the curve*

- Symbolic ODEs
  - algebraic solutions,

Baldassari & Dwork 79, *On second order linear differential equations with algebraic solutions*

  - Galois group

Compoint & Singer 79, *Relations linéaires entre solutions d'équations différentielles*

Physics

- Quasi periodic solutions of the KP equation
- Special solutions of the KdV and NLS equations

Deconinck & Segur 1998, *The KP equation with quasiperiodic initial data*

Deconinck & Patterson 2007, *Computing the Abel map*

Computing the Abel map: overview

Patterson 2007, *Algebro-geometric algorithms for integrable systems*

1. Computing the period matrix

   van Hoeij & Deconinck 2001, *Computing Riemann matrices of algebraic curves*

   - Monodromy computation,
   - Use it to compute the homology basis,

     $\Longrightarrow$ Tretkoff[2] 1984, *Combinatorial Group Theory, Riemann Surfaces and Differential Equations.*
   - Compute a basis of holomorphic differential on the surface,
   - Compute the periods.
2. Make analytic continuation along the needed paths.

Regular and critical points: a projective point of view

$$\mathcal{C} = \{(x, y) \in \mathbb{C}^2 \mid F(x, y) = 0\}$$

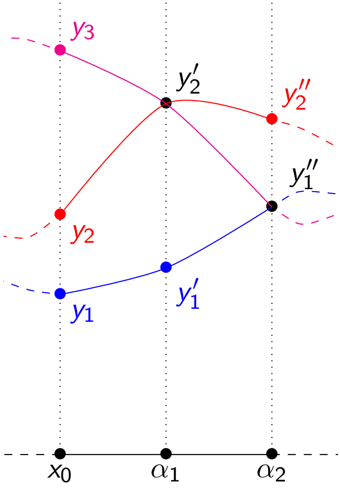
Let $x_0 \in \mathbb{C}$:

- **Fiber** at x_0:

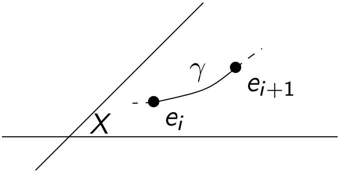
  $\mathcal{F}(x_0) = \{\text{roots of } F(x_0, Y) = 0\}$.

- **Regular point** : $\#\mathcal{F}(x_0) = d_Y$.
- **Critical point** : $\#\mathcal{F}(x_0) < d_Y$.

  $\Longrightarrow$ roots of $R_F = \mathtt{Res}_Y(F, F_Y)$

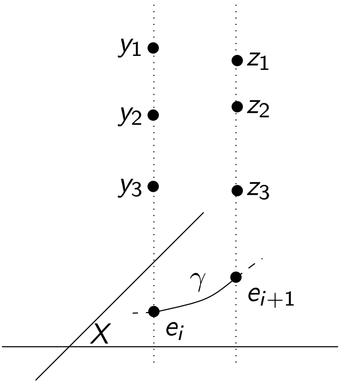


Analytic continuation : compute fibers and connect



1. Choice of paths,
2. Choice of connection points,

Analytic continuation : compute fibers and connect



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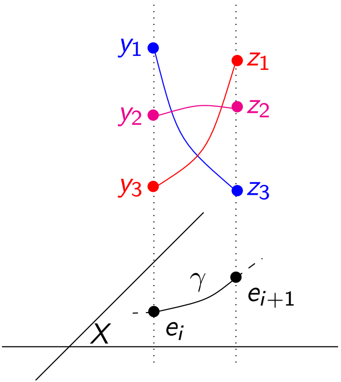
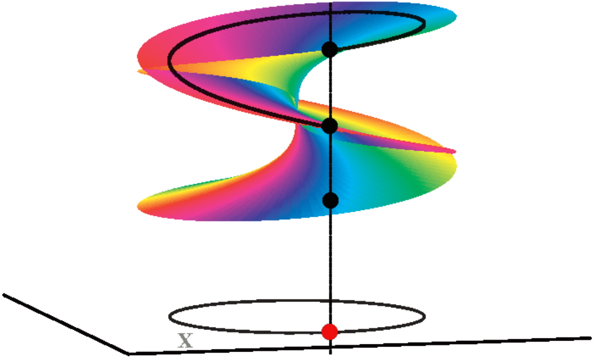
Analytic continuation : compute fibers and connect



1. Choice of paths,
2. Choice of connection points,
3. Compute the fibers,
4. Connection method.

Local monodromy: $F(X, Y) = Y^3 - X$ around $X = 0$

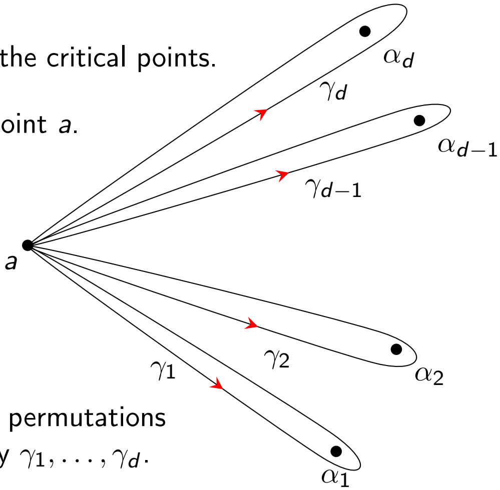


- analytic continuation around the critical point $X = 0$

  $\Longrightarrow$ permutation on the fiber

Monodromy group

- Let $\alpha_1, \ldots, \alpha_d$ denote the critical points.
- We fix a regular base point a.



- We will compute the d permutations $\sigma_1, \ldots, \sigma_d$ generated by $\gamma_1, \ldots, \gamma_d$.
- These permutations generate the monodromy group.

Computing the Abel map: details

Patterson 2007, *Algebro-geometric algorithms for integrable systems*

1. Computing the period matrix

   van Hoeij & Deconinck 2001, *Computing Riemann matrices of algebraic curves*

   - Monodromy computation,
   - Use it to compute the homology basis,
   - Compute a basis of holomorphic differential on the surface,
   - Compute the periods.

2. Make analytic continuation along the needed paths.
   - Makes analytic continuation using first derivative only,
   - Uses Puiseux expansion when $\mathcal{P}$ is close from a critical point.

Connexion method

- van Hoeij & Deconinck 2001
  - Connexion using only derivatives (order 1),
  - No certification that the connexion is correct.
- van Hoeij & Rybowicz 2006
  - Certified algorithm using order 1 truncations,
  - Efficiency problem: sometimes too many intermediary steps (two close critical points).
- Poteaux 2007, *Computing monodromy groups defined by plane algebraic curves*
  - Power series expansions at well chosen truncation orders,
  - Trade-off between truncations orders and the number of steps,
  - (similar to Chudnovsky^2 86, 87, 90, van der Hoeven 99 for ODEs)
  - *useful for the integration part*.

Choice of the paths

- Most methods: going as far as possible from the critical points (one loop = *17 steps*)

  van Hoeij & Deconinck 01, Chudnovsky^2 86, 87, 90, van der Hoeven 99

  - numerical problems when two critical points are very close...

- Another way : computing series *above critical points*
  - van der Hoeven 01 (ODE) Poteaux 07 (Puiseux series)
  - one loop = one evaluation,
  - Puiseux series are also useful for the Abel map !

    → needed when the point $\mathcal{P}$ is close to a critical point,

    Deconinck-Patterson 2007, *Computing the Abel map*

    → it enables to compute integral basis (thus cohomology).

    van Hoeij 1994, *an algorithm for computing an integral basis in an algebraic function field*

Puiseux series

- A generalization of power series:
  - above a regular point: $Y(X) = \sum_{k=0}^{\infty} \beta_k (X - x_0)^k$
  - above a critical point: $Y(X) = \sum_{k=n}^{\infty} \beta_k (X - x_0)^{k/e}$
- Costly to compute:
  - coefficient growth and high degree extension field
  - one example: $(Y^3 - X)((Y-1)^2 - X)(Y - 2 - X^2) + X^2 Y^5$

    $\Longrightarrow$ extension of degree 92, rational numbers with 132 digits
  - Bit complexity: $\tilde{\mathcal{O}}({d_Y}^{32}{d_X}^4 \mathrm{ht}(F)^2)$ Walsh 2000
  - Best known arithmetic complexity: $\tilde{\mathcal{O}}(D^5)$ Poteaux 2008

Computing Puiseux series efficiently

- A modular-numeric strategy to avoid coefficient growth:
  - First computation modulo a *well-chosen* prime number

    $\Longrightarrow$ structure of the solution,
  - Using it, conduct numerical computations to get an approximation of the coefficients.
- Towards a faster algorithm
  - Computing fast "Taylor parts" + factorize F in $\mathbb{K}[[X]][Y]$ during the process + using optimal truncation bounds via relaxed algorithms $\Longrightarrow \tilde{\mathcal{O}}(D^4)$
  - Being able to factorize $F = F_1 \cdot F_2$ in $\mathbb{K}[[X]][Y]$ knowing singular part of Puiseux series of $F_1 \Longrightarrow \tilde{\mathcal{O}}(D^3)$

What to be done yet ?

- Numerical part of Puiseux is not certified and might be improved,
- Fast Puiseux algorithm to be finished and written properly,
- Still one missing point for the $\tilde{O}(D^3)$ strategy,
- Efficient implementation of all these algorithms,
- Studying the next parts of the Abel map (optimizations, implementation. . .)